

QUOTIENTS OF ULTRAGRAPH C^* -ALGEBRAS

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ABSTRACT. Let \mathcal{G} be an ultragraph and $C^*(\mathcal{G})$ be its C^* -algebra defined by Tomforde. If $I_{(H,B)}$ is a gauge invariant ideal of $C^*(\mathcal{G})$, we investigate structure of the quotient $C^*(\mathcal{G})/I_{(H,B)}$ by introducing the notion of quotient ultragraph $\mathcal{G}/(H,B)$ and its C^* -algebra $C^*(\mathcal{G}/(H,B))$. We prove two kinds of uniqueness theorems, the gauge invariant and the Cuntz-Krieger ones, for $C^*(\mathcal{G}/(H,B))$ and show that $C^*(\mathcal{G}/(H,B)) \cong C^*(\mathcal{G})/I_{(H,B)}$. Then we characterize gauge invariant ideals as well as primitive gauge invariant ideals of $C^*(\mathcal{G}/(H,B))$ which are corresponding with specific ideals of $C^*(\mathcal{G})$. In particular, we determine primitive gauge invariant ideals of an ultragraph C^* -algebra $C^*(\mathcal{G})$. Furthermore, we define condition (K) for a quotient ultragraph $\mathcal{G}/(H,B)$ and we show under this condition that all ideals of $C^*(\mathcal{G}/(H,B))$ are gauge invariant and the real rank of $C^*(\mathcal{G}/(H,B))$ is zero.

1. INTRODUCTION

In order to bring graph C^* -algebras and Exel-Laca algebras together under one theory, Tomforde introduced in [17] the notion of an ultragraph and an associated C^* -algebra. Ultragraphs are defined similar to (directed) graphs in which the range of each edge is allowed to be a nonempty set of vertices rather than a single vertex. However, the class of ultragraph C^* -algebras are strictly larger than graph C^* -algebras as well as Exel-Laca algebras (see [18, Section 5]). Due to the similarity, parts of fundamental results for graph C^* -algebras, such as the Cuntz-Krieger and the gauge invariant uniqueness theorems, simplicity, ideal structure, and K -theory computation have been extended to the setting of ultragraphs.

By constructing an specific topological quiver $\mathcal{Q}(\mathcal{G})$ from an ultragraph \mathcal{G} , Katsura et al. investigated the properties of ultragraph C^* -algebras using that of topological quivers [13]. In particular, they showed that every gauge invariant ideal of the C^* -algebra $C^*(\mathcal{G})$ is of the form $I_{(H,B)}$ corresponding to an admissible pair (H,B) in \mathcal{G} . But the graph C^* -algebras (and also the ultragraph C^* -algebras) are a very trivial case of the topological quiver algebras [14] in which the sets of vertices and edges are considered with discrete topology and the Radon measures are special counting measures.

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So, it seems that the structure of ultragraph C^* -algebras is more closer to that of graph C^* -algebras than topological quiver algebras.

Associated to a gauge invariant ideal $I_{(H,B)}$ in a graph C^* -algebra $C^*(E)$, there is a (quotient) graph $E/(H, B)$ such that $C^*(E)/I_{(H,B)} \cong C^*(E/(H, B))$ (see [1, 5]). So, the class of graph C^* -algebras includes such their quotients and the results and properties of graph C^* -algebras could be applied for them. For examples, the contexts as simplicity, K -theory, primitivity, and topological stable rank are directly related to the structure of ideals and quotients. But one of important difficulties to study ultragraph C^* -algebras comparing with graph C^* -algebras is due to structure of their quotients; if $I_{(H,B)}$ is a gauge invariant ideal in an ultragraph C^* -algebra $C^*(\mathcal{G})$, the quotient algebra $C^*(\mathcal{G})/I_{(H,B)}$ is not known to be realized as an ultragraph C^* -algebras. This causes that we could not apply many graph C^* -algebraic methods for the ultragraph C^* -algebras.

The aim of this article is to give a way for considering quotients of ultragraph C^* -algebras as a kind of universal C^* -algebras which are similar to ultragraph C^* -algebras. For this, associated to any admissible pair (H, B) in an ultragraph \mathcal{G} , we introduce a notion of quotient ultragraph $\mathcal{G}/(H, B)$ and its relative universal C^* -algebra $C^*(\mathcal{G}/(H, B))$ such that $C^*(\mathcal{G})/I_{(H,B)} \cong C^*(\mathcal{G}/(H, B))$. We shall see that the structure of our quotient ultragraph C^* -algebras is close to that of ultragraph C^* -algebras and we could prove the gauge invariant and the Cuntz-Krieger uniqueness theorems for them. The uniqueness theorems help us to show when a representation of these algebras is injective. Since the properties of a quotient ultragraph $\mathcal{G}/(H, B)$ come from its initial ultragraph \mathcal{G} respect to (H, B) , it seems that working with quotient ultragraphs is much more easier than with topological quivers in [13]. Also by quotient ultragraphs, many traditional graph C^* -algebra techniques may be extended to the ultragraph setting with some small modifications. The initial idea for defining quotient ultragraphs comes from [9] where quotient labelled graphs are introduced. However, many main results of this paper are not known for labelled graphs. Also, any restricted assumption such as set-finiteness or receiver set-finiteness does not be supposed here for ultragraphs.

This article is organized as follows. We begin in Section 2 by giving some definitions and preliminaries about the ultragraphs and their C^* -algebras which will be used in the next sections. In Section 3, for any admissible pair (H, B) in an ultragraph \mathcal{G} we introduce quotient ultragraph $\mathcal{G}/(H, B)$ and associate a universal C^* -algebra $C^*(\mathcal{G}/(H, B))$ to it. For this, the ultragraph \mathcal{G} is replaced by an extended ultragraph $\overline{\mathcal{G}}$ and similar to [9, Section 3], we define an equivalent relation \sim on $\overline{\mathcal{G}}$. The quotient ultragraph $\mathcal{G}/(H, B)$ is the ultragraph $\overline{\mathcal{G}}$ with equivalent classes $\{[A] : A \in \overline{\mathcal{G}}^0\}$. Then we associate to it a universal C^* -algebra $C^*(\mathcal{G}/(H, B))$ so that $C^*(\mathcal{G}/(H, B)) \cong C^*(\mathcal{G})/I_{(H,B)}$ by Proposition 5.1. In Section 4, by approaching with graph C^* -algebras, the

gauge invariant and the Cuntz-Krieger uniqueness theorems will be proved for our quotient ultragraphs C^* -algebras.

Section 5 is devoted to the structure of gauge invariant ideals in $C^*(\mathcal{G}/(H, B))$. We first prove that $C^*(\mathcal{G}/(H, B))$ is isometrically isomorphic to the quotient of $C^*(\mathcal{G})$ by $I_{(H, B)}$. Then we see that every gauge invariant ideal $I_{(K, S)}$ in $C^*(\mathcal{G})$ with $H \subseteq K$, $B \subseteq K \cup S$ induces an ideal $J_{[K, S]}$ in $C^*(\mathcal{G}/(H, B))$ and all gauge invariant ideals of $C^*(\mathcal{G}/(H, B))$ are of this form. In particular in Theorem 5.3 it is shown that $C^*(\mathcal{G}/(H, B))/J_{[K, S]} \cong C^*(\mathcal{G}/(K, S))$. In Section 6 and 7, we see that some graph C^* -algebra methods may be applied for the C^* -algebras of ultragraphs and quotient ultragraphs which could not before. In Section 6, Condition (K) is defined for a quotient ultragraph $\mathcal{G}/(H, B)$ so that all ideals of $C^*(\mathcal{G}/(H, B))$ to be gauge invariant. Also, we show that $\mathcal{G}/(H, B)$ satisfies Condition (K) if and only if the real rank of the C^* -algebra $C^*(\mathcal{G}/(H, B))$ is zero. As a corollary, we can recover both [13, Proposition 7.3] and [12, Proposition 5.26] by quotient ultragraphs whereas our method is quite different from those of [13] and [12]. In Section 7, we characterize primitive gauge invariant ideals of ultragraph C^* -algebras and we can describe primitive gauge invariant ideals of the C^* -algebra of quotient ultragraphs as well.

2. PRELIMINARIES

In this section, we review basic definitions and some properties of ultragraph C^* -algebras which will be needed through the paper. For more details, we refer the reader to [17] and [13].

Definition 2.1. An *ultragraph* is a quadruple $\mathcal{G} = (G^0, \mathcal{G}^1, r_{\mathcal{G}}, s_{\mathcal{G}})$ consisting of a countable vertex set G^0 , a countable edge set \mathcal{G}^1 , the source map $s_{\mathcal{G}} : \mathcal{G}^1 \rightarrow G^0$, and the range map $r_{\mathcal{G}} : \mathcal{G}^1 \rightarrow \mathcal{P}(G^0) \setminus \{\emptyset\}$, where $\mathcal{P}(G^0)$ is the collection of all subsets of G^0 . If $r_{\mathcal{G}}(e)$ is a singleton vertex for each edge $e \in \mathcal{G}^1$, then \mathcal{G} is an ordinary (directed) graph.

For our convenience, we use the notation \mathcal{G}^0 of [13] rather than [17, 18]. For any set X , a nonempty family \mathcal{C} of the elements of $\mathcal{P}(X)$ is said an *algebra* if it is closed under the set operations \cap , \cup , and \setminus . If \mathcal{G} is an ultragraph, the smallest algebra in $\mathcal{P}(G^0)$ containing $\{\{v\} : v \in G^0\}$ and $\{r_{\mathcal{G}}(e) : e \in \mathcal{G}^1\}$ is denoted by \mathcal{G}^0 . We simply denote every singleton set $\{v\}$ by v . So, G^0 could be considered as a subset of \mathcal{G}^0 .

Definition 2.2. For each $n \geq 1$, a *path* α of length $|\alpha| = n$ in \mathcal{G} is a sequence $\alpha = e_1 \dots e_n$ of edges such that $s(e_{i+1}) \in r(e_i)$ for $1 \leq i \leq n-1$. If also $s(e_1) \in r(e_n)$, α is called a *loop* or a *closed path*. We write α^0 for the set $\{s_{\mathcal{G}}(e_i) : 1 \leq i \leq n\}$. The elements of \mathcal{G}^0 is said paths of length zero. The set of all paths in \mathcal{G} is denoted by \mathcal{G}^* . We may naturally extend the maps $s_{\mathcal{G}}, r_{\mathcal{G}}$ on \mathcal{G}^* by defining $s_{\mathcal{G}}(A) = r_{\mathcal{G}}(A) = A$ for $A \in \mathcal{G}^0$, and $r_{\mathcal{G}}(\alpha) = r_{\mathcal{G}}(e_n)$, $s_{\mathcal{G}}(\alpha) = s_{\mathcal{G}}(e_1)$ for every path $\alpha = e_1 \dots e_n$.

Definition 2.3. Let \mathcal{G} be an ultragraph. A *Cuntz-Krieger \mathcal{G} -family* is a set of partial isometries $\{s_e : e \in \mathcal{G}^1\}$ with mutually orthogonal ranges and a set of projections $\{p_A : A \in \mathcal{G}^0\}$ that satisfy

- (1) $p_\emptyset = 0$, $p_A p_B = p_{A \cap B}$, and $p_{A \cup B} = p_A + p_B - p_{A \cap B}$ for all $A, B \in \mathcal{G}^0$,
- (2) $s_e^* s_e = p_{r_{\mathcal{G}}(e)}$ for $e \in \mathcal{G}^1$,
- (3) $s_e s_e^* \leq p_{s_{\mathcal{G}}(e)}$ for $e \in \mathcal{G}^1$, and
- (4) $p_v = \sum_{s_{\mathcal{G}}(e)=v} s_e s_e^*$ whenever $0 < |s_{\mathcal{G}}^{-1}(v)| < \infty$.

The C^* -algebra $C^*(\mathcal{G})$ of \mathcal{G} is the (unique) C^* -algebra generated by a universal Cuntz-Krieger \mathcal{G} -family.

As noted in [17, Remark 2.13], we have

$$C^*(\mathcal{G}) = \overline{\text{span}} \{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{G}^*, A \in \mathcal{G}^0, \text{ and } r_{\mathcal{G}}(\alpha) \cap r_{\mathcal{G}}(\beta) \cap A \neq \emptyset\},$$

where $s_\alpha := s_{e_1} \dots s_{e_n}$ if $\alpha = e_1 \dots e_n$ and $s_\alpha := p_A$ if $\alpha = A$.

Remark 2.4. As noted in [17, Section ...], every graph C^* -algebra is an ultragraph C^* -algebra. If $E = (E^0, E^1, r_E, s_E)$ is a graph, a collection $\{s_e, p_v : v \in E^0, e \in E^1\}$ containing mutually orthogonal projections p_v and partial isometries s_e is a Cuntz-Krieger E -family if:

- (1) $s_e^* s_e = p_{r_E(e)}$ for $e \in E^1$,
- (2) $s_e s_e^* \leq p_{s_E(e)}$ for $e \in E^1$, and
- (3) $p_v = \sum_{s_E(e)=v} s_e s_e^*$ whenever $0 < |s_E^{-1}(v)| < \infty$.

Then the C^* -algebra $C^*(E)$ is generated by a universal Cuntz-Krieger E -family.

By the universal property, $C^*(\mathcal{G})$ admits the *gauge action* of the unit circle \mathbb{T} . By an *ideal*, we mean a closed two sided one. Using the known results for the C^* -algebra of topological quivers [13], the structure of gauge invariant ideals of $C^*(\mathcal{G})$ was described in [13, Theorem 6.12] via a one-to-one correspondence with the admissible pairs of \mathcal{G} .

Definition 2.5. A subset $H \subseteq \mathcal{G}^0$ is said to be *hereditary* if H satisfies the following:

- (1) $s_{\mathcal{G}}(e) \in H$ implies $r_{\mathcal{G}}(e) \in H$ for all $e \in \mathcal{G}^1$,
- (2) $A \cup B \in H$ for all $A, B \in H$, and
- (3) if $A \in H$, $B \in \mathcal{G}^0$, and $B \subseteq A$, then $B \in H$.

Also, a subset $H \subseteq \mathcal{G}^0$ is called to be *saturated* if for any $v \in \mathcal{G}^0$ with $0 < |s_{\mathcal{G}}^{-1}(v)| < \infty$, then $r_{\mathcal{G}}(s_{\mathcal{G}}^{-1}(v)) \subseteq H$ implies $v \in H$. The *saturated hereditary closure* of a subset $H \subseteq \mathcal{G}^0$ is the smallest hereditary and saturated subset \overline{H} of \mathcal{G}^0 containing H .

Let H be a saturated hereditary subset of \mathcal{G}^0 . The *breaking vertices* of H is denoted by

$$B_H := \{w \in \mathcal{G}^0 : |s_{\mathcal{G}}^{-1}(w)| = \infty \text{ but } 0 < |r_{\mathcal{G}}(s_{\mathcal{G}}^{-1}(w)) \cap (\mathcal{G}^0 \setminus H)| < \infty\}.$$

If H is a saturated hereditary subset of \mathcal{G}^0 and $B \subseteq B_H$, we say (H, B) to be an *admissible pair* in \mathcal{G} . For an admissible pair (H, B) in \mathcal{G}^0 , we define the ideal $I_{(H, B)}$ of $C^*(\mathcal{G})$ generated by

$$\{p_A : A \in \mathcal{G}^0\} \cup \{p_w^H : w \in B\},$$

where $p_w^H := p_w - \sum_{s_{\mathcal{G}}(w)=e, r_{\mathcal{G}}(w) \notin H} s_e s_e^*$. It is clear that each ideal $I_{(H, B)}$ is gauge invariant and [11, Theorem 6.12] implies that every gauge invariant ideal I of $C^*(\mathcal{G})$ is of the form $I_{(H, B)}$ if

$$H = \{A : p_A \in I\} \text{ and } B = \{w \in B_H : p_w^H \in I\}.$$

3. QUOTIENT ULTRAGRAPH C^* -ALGEBRAS

In this section, corresponding a quotient C^* -algebra $C^*(\mathcal{G})/I_{(H, B)}$, we introduce the notion of a quotient ultragraph $\mathcal{G}/(H, B)$ and its relative universal C^* -algebra $C^*(\mathcal{G}/(H, B))$ such that $C^*(\mathcal{G})/I_{(H, B)} \cong C^*(\mathcal{G}/(H, B))$. So, let us fix an ultragraph \mathcal{G} and an admissible pair (H, B) in \mathcal{G} and suppose that $C^*(\mathcal{G}) = C^*(s_e, p_A)$. Since the ideal $I_{(H, B)}$ is generated by $\{p_A : A \in H\} \cup \{p_w^H : w \in B\}$, a quotient $C^*(\mathcal{G})/I_{(H, B)}$ may have some generators more than the projections $\{p_A + I_{(H, B)} : A \in \mathcal{G}^0 \setminus H\}$. So, we first extend \mathcal{G} to $\overline{\mathcal{G}}$ corresponding with $B_H \setminus B$.

Suppose that $\mathcal{G} = (G^0, \mathcal{G}^0, r_{\mathcal{G}}, s_{\mathcal{G}})$ and add the vertices $\{w' : w \in B_H \setminus B\}$ to G^0 . For each $A \in \mathcal{G}^0$, denote $\overline{A} := A \cup \{w' : w \in A \cap (B_H \setminus B)\}$ and define the maps s', r' as $s'(e) := s_{\mathcal{G}}(e)$ and $r'(e) := \overline{r_{\mathcal{G}}(e)}$ for every $e \in \mathcal{G}^1$. For constructing an appropriate quotient ultragraph associated to (H, B) , we consider the ultragraph $\overline{\mathcal{G}} = (\overline{G}^0, \overline{\mathcal{G}}^1, r', s')$ instead of \mathcal{G} , where

$$\overline{G}^0 := G^0 \cup \{w' : w \in B_H \setminus B\} \text{ and } \overline{\mathcal{G}}^1 := \mathcal{G}^1.$$

As usual, we write $\overline{\mathcal{G}}^0$ for the algebra generated by $\overline{G}^0 \cup \{r'(e) : e \in \overline{\mathcal{G}}^1\}$. Note that the set H is also a saturated hereditary subset of $\overline{\mathcal{G}}^0$ and for any $A \in H$, we have $\overline{A} = A$ as $A \cap B_H = \emptyset$.

Remark 3.1. As $\overline{\mathcal{G}}^0$ is an algebra generated by $\{v, w', r'(e) : v \in G^0, w \in B_H \setminus B, e \in \overline{\mathcal{G}}^1\}$, where each w' is a sink in $\overline{\mathcal{G}}$, if a collection $\{P_v, P_{\overline{A}}, S_e : v \in G^0, A \in \mathcal{G}^0, e \in \overline{\mathcal{G}}^1\}$ satisfies the conditions (1)-(4) of Definition (...) associated to $\overline{\mathcal{G}}$, then it induces a Cuntz-Krieger $\overline{\mathcal{G}}$ -family $\{S_e, P_A : A \in \overline{\mathcal{G}}^0, e \in \overline{\mathcal{G}}^1\}$ by defining

$$\begin{aligned} P_{A \cap B} &:= P_A P_B \\ P_{A \cup B} &:= P_A + P_B - P_A P_B \\ P_{A \setminus B} &:= P_A - P_A P_B \end{aligned}$$

for other set projections.

The following proposition shows that the C^* -algebras of \mathcal{G} and $\overline{\mathcal{G}}$ are equal.

Proposition 3.2. *Let \mathcal{G} be an ultragraph and (H, B) be an admissible pair in \mathcal{G} . If $\overline{\mathcal{G}}$ is the extended ultragraph associated to (H, B) as above, then $C^*(\mathcal{G}) \cong C^*(\overline{\mathcal{G}})$.*

Proof. Suppose that $C^*(\mathcal{G}) = C^*(t_e, q_A)$ and $C^*(\overline{\mathcal{G}}) \cong (s_e, p_C)$. If we define

$$\begin{aligned} P_v &:= q_v && \text{for } v \in G^0 \setminus (B_H \setminus B) \\ P_w &:= \sum_{\substack{s_{\mathcal{G}}(e)=w \\ r_{\mathcal{G}}(e) \notin H}} t_e t_e^* && \text{for } w \in B_H \setminus B \\ P_{w'} &:= q_w - \sum_{\substack{s_{\mathcal{G}}(e)=w \\ r_{\mathcal{G}}(e) \notin H}} t_e t_e^* && \text{for } w \in B_H \setminus B \\ P_{\overline{A}} &:= q_A && \text{for } \overline{A} \in \overline{\mathcal{G}}^0 \\ S_e &:= t_e && \text{for } e \in \overline{\mathcal{G}}^1 \end{aligned}$$

then the family $\{P_v, P_w, P_{w'}, P_{\overline{A}}, S_e\}$ induces a Cuntz-Krieger $\overline{\mathcal{G}}$ -family in $C^*(\mathcal{G})$. Since all vertex projections of this family are nonzero (that follows all set projections are nonzero), the gauge-invariant uniqueness theorem [17, Theorem 6.8] implies that the $*$ -homomorphism $\phi : C^*(\overline{\mathcal{G}}) \rightarrow C^*(\mathcal{G})$ with $\phi(p_*) = P_*$ and $\phi(s_*) = S_*$ is injective. Since the family generates all $C^*(\mathcal{G})$, we conclude that ϕ is an isomorphism. \square

For defining a quotient ultragraph of \mathcal{G} associated to (H, B) , we use the following equivalent relation on $\overline{\mathcal{G}}$ which is corresponding with the equivalent classes in the quotient $C^*(\mathcal{G})/I_{(H,B)}$. We then show that the algebraic operations in $\overline{\mathcal{G}}$ are invariant under this relation.

Definition 3.3. Let (H, B) be an admissible pair in \mathcal{G} and $\overline{\mathcal{G}}$ be the above ultragraph. We define the relation \sim on $\overline{\mathcal{G}}^0$ as

$$A \sim B \iff \exists V \in H \text{ such that } A \cup V = B \cup V.$$

Note that $A \sim B$ if and only if both $A \setminus B, B \setminus A$ belong to H .

Lemma 3.4. *The relation of Definition 3.3 is an equivalent relation on $\overline{\mathcal{G}}^0$. Furthermore, the operations*

$$[A] \cup [B] := [A \cup B], \quad [A] \cap [B] := [A \cap B], \quad \text{and} \quad [A] \setminus [B] := [A \setminus B]$$

are well-defined on the equivalent classes $\{[A] : A \in \overline{\mathcal{G}}^0\}$.

Proof. It is straightforward to verify that the relation \sim is reflexive, symmetric, and associative. So, we prove the second assertion. Let $[A_1] = [A_2]$, $[B_1] = [B_2]$, and $A_1 \cup V = A_2 \cup V$, $B_1 \cup W = B_2 \cup W$ for some $V, W \in H$. Since $V \cup W \in H$ by the hereditary property, we have

$$\begin{aligned} (A_1 \cup B_1) \cup (V \cup W) &= (A_2 \cup B_2) \cup (V \cup W) \\ \implies [A_1 \cup B_1] &= [A_2 \cup B_2], \end{aligned}$$

and also,

$$\begin{aligned} (A_1 \cap B_1) \cup (V \cup W) &= (A_1 \cup (V \cup W)) \cap (B_1 \cup (V \cup W)) \\ &= (A_2 \cup (V \cup W)) \cap (B_2 \cup (V \cup W)) \end{aligned}$$

$$\begin{aligned} &= (A_2 \cap B_2) \cup (V \cup W) \\ \implies [A_1 \cap B_1] &= [A_2 \cap B_2] \end{aligned}$$

For the third operation, since $B_1 \cup W = B_2 \cup W$, we have $B_1 \setminus B_2, B_2 \setminus B_1 \subseteq W$. Then

$$\begin{aligned} (A_1 \setminus B_1) \cup W &= (A_1 \setminus ((B_1 \cap B_2) \cup (B_1 \setminus B_2))) \cup W \\ &= (A_1 \setminus (B_1 \cap B_2)) \cup W \end{aligned}$$

and similarly,

$$(A_2 \setminus B_2) \cup W = (A_2 \setminus (B_1 \cap B_2)) \cup W.$$

Now the fact $A_1 \cup V = A_2 \cup V$ yields that

$$\begin{aligned} (A_1 \setminus B_1) \cup (V \cup W) &= (A_1 \setminus (B_1 \cap B_2)) \cup (V \cup W) \\ &= ((A_1 \cup V) \setminus (B_1 \cap B_2)) \cup (V \cup W) \\ &= ((A_2 \cup V) \setminus (B_1 \cap B_2)) \cup (V \cup W) \\ &= (A_2 \setminus B_2) \cup (V \cup W), \end{aligned}$$

and hence, $[A_1 \setminus B_1] = [A_2 \setminus B_2]$. Consequently, the operations are well-defined. \square

By this relation we define our quotient ultragraph.

Definition 3.5. Let \mathcal{G} be an ultragraph, (H, B) be an admissible pair in \mathcal{G} , and consider the equivalent relation of Definition 3.3 in $\overline{\mathcal{G}}$. The *quotient ultragraph of \mathcal{G} by (H, B)* is the quintuple $\mathcal{G}/(H, B) = (\Phi(G^0), \Phi(\mathcal{G}^0), \Phi(\mathcal{G}^1), r, s)$, where

$$\begin{aligned} \Phi(G^0) &:= \{[v] : v \in G^0 \setminus H\} \cup \{[w'] : w \in B_H \setminus B\}, \\ \Phi(\mathcal{G}^0) &:= \{[A] : A \in \overline{\mathcal{G}}^0\}, \\ \Phi(\mathcal{G}^1) &:= \{e \in \overline{\mathcal{G}}^1 : r'(e) \notin H\}, \end{aligned}$$

and $r : \Phi(\mathcal{G}^1) \rightarrow \Phi(\mathcal{G}^0)$, $s : \Phi(\mathcal{G}^1) \rightarrow \Phi(G^0)$ are the maps defined by

$$s(e) := [s'(e)] \quad \text{and} \quad r(e) = [r'(e)].$$

We refer to $\Phi(G^0)$ as the vertices of $\mathcal{G}/(H, B)$.

Remark 3.6. Lemma 3.4 implies that $\Phi(\mathcal{G}^0)$ is the smallest algebra containing

$$\{[v], [w'] : v \in G^0 \setminus H, w \in B_H \setminus B\} \cup \{[r'(e)] : e \in \overline{\mathcal{G}}^1\}.$$

Also, for every $[v] \in \Phi(G^0)$, we have $[v] \neq [\emptyset]$ as $v \notin H$.

Notation 3.7. We usually denote $[v]$ instead of $[\{v\}]$. Also, if $[A] \cap [B] = [A]$ we write $[A] \subseteq [B]$.

Now we introduce representations of quotient ultragraphs and their relative C^* -algebras.

Definition 3.8. Let $\mathcal{G}/(H, B)$ be a quotient ultragraph. A representation of $\mathcal{G}/(H, B)$ is a set of partial isometries $\{T_e : e \in \Phi(\mathcal{G}^1)\}$ and a set of projections $\{Q_{[A]} : [A] \in \Phi(\mathcal{G}^0)\}$ that satisfy

- (1) $Q_{[\emptyset]} = 0$, $Q_{[A \cap B]} = Q_{[A]}Q_{[B]}$, and $Q_{[A \cup B]} = Q_{[A]} + Q_{[B]} - Q_{[A \cap B]}$,
- (2) $T_e^*T_e = Q_{r(e)}$ and $T_e^*T_f = 0$ when $e \neq f$,
- (3) $T_eT_e^* \leq Q_{s(e)}$, and
- (4) $Q_{[v]} = \sum_{s(e)=[v]} T_eT_e^*$, whenever $0 < |s^{-1}([v])| < \infty$.

The C^* -algebra $C^*(\mathcal{G}/(H, B))$ of $\mathcal{G}/(H, B)$ is the C^* -algebra generated by a universal representation $\{t_e, q_{[A]} : [A] \in \Phi(\mathcal{G}^0), e \in \Phi(\mathcal{G}^1)\}$ which exists by Theorem 3.11 below..

Remark 3.9. Recall that the universality of $C^*(\mathcal{G}/(H, B))$ means that if $\{T_e, Q_{[A]}\}$ is a representation of $\mathcal{G}/(H, B)$ in a C^* -algebra X , then there exists a $*$ -homomorphism $\phi : C^*(\mathcal{G}/(H, B)) \rightarrow X$ such that $\phi(q_{[A]}) = Q_{[A]}$ and $\phi(t_e) = T_e$ for all $[A] \in \Phi(\mathcal{G}^0)$ and $e \in \Phi(\mathcal{G}^1)$.

Note that if $\alpha = e_1 \dots e_n$ is a path in $\overline{\mathcal{G}}$ with $r'(\alpha) \notin H$, then the hereditary property of H yields $r'(e_i) \notin H$, and so $e_i \in \Phi(\mathcal{G}^1)$ for all $1 \leq i \leq n$. In this case we denote $t_\alpha := t_{e_1} \dots t_{e_n}$. As for ultragraphs we set

$$(\mathcal{G}/(H, B))^* := \{[A] : [A] \neq [\emptyset]\} \cup \{\alpha \in \overline{\mathcal{G}}^* : r(\alpha) \neq [\emptyset]\}$$

and we could extend the maps s, r on $(\mathcal{G}/(H, B))^*$ by defining

$$s([A]) = r([A]) = [A] \text{ and } s(\alpha) = s(e_1), \quad r(\alpha) = r(e_n).$$

The proof of next lemma is similar to the argument of [17, Lemmas 2.8 and 2.9].

Lemma 3.10. *Let $\mathcal{G}/(H, B)$ be a quotient ultragraph and $\{T_e, Q_{[A]}\}$ be a representation of $\mathcal{G}/(H, B)$. Then any nonzero word in T_e , $Q_{[A]}$, and T_f^* may be written as a finite linear combination of the forms $T_\alpha Q_{[A]} T_\beta^*$ for some $\alpha, \beta \in (\mathcal{G}/(H, B))^*$ and $[A] \in \Phi(\mathcal{G}^0)$ with $[A] \cap r(\alpha) \cap r(\beta) \neq [\emptyset]$.*

Theorem 3.11. *Let $\mathcal{G}/(H, B)$ be a quotient ultragraph. Then there exists a (unique) C^* -algebra $C^*(\mathcal{G}/(H, B))$ generated by a universal representation $\{t_e, q_{[A]} : [A] \in \Phi(\mathcal{G}^0), e \in \Phi(\mathcal{G}^1)\}$ for $\mathcal{G}/(H, B)$. Furthermore, the t_e 's and $q_{[A]}$'s are all nonzero for $[A] \neq [\emptyset]$, $e \in \Phi(\mathcal{G}^1)$.*

Proof. An standard argument similar to the proof of [17, Theorem 2.11] shows the existence of a universal $C^*(\mathcal{G}/(H, B))$. Also, the universality implies that such C^* -algebra is unique up to isomorphism. So, we show the second assertion. Suppose that $C^*(\overline{\mathcal{G}}) = C^*(s_e, p_A)$ and consider $I_{(H, B)}$ as an ideal in $C^*(\overline{\mathcal{G}})$ by the isomorphism of Proposition 3.2. Define

$$\begin{aligned} Q_{[A]} &:= p_A + I_{(H, B)} \\ T_e &:= s_e + I_{(H, B)}. \end{aligned}$$

for every $A \in \overline{\mathcal{G}}^0$ and $e \in \overline{\mathcal{G}}^1$. Note that if $A_1 \cup V = A_2 \cup V$ for some $V \in H$, then $p_{A_1} + p_{V \setminus A_1} = p_{A_2} + p_{V \setminus A_2}$ and we have $p_{A_1} + I_{(H,B)} = p_{A_2} + I_{(H,B)}$ as $V \setminus A_1, V \setminus A_2 \in H$. So, the definition of $Q_{[A]}$'s is well-defined. It is straightforward to verify that the family $\{T_e, Q_{[A]} : [A] \in \Phi(\mathcal{G}^0), e \in \Phi(\mathcal{G}^1)\}$ is a representation for $\mathcal{G}/(H, B)$ in $C^*(\overline{\mathcal{G}})/I_{(H,B)}$ and by the universality, there is a $*$ -homomorphism $\phi : C^*(\mathcal{G}/(H, B)) \rightarrow C^*(\overline{\mathcal{G}})/I_{(H,B)}$ such that $\phi(t_e) = T_e$ and $\phi(q_{[A]}) = Q_{[A]}$, for $[A] \in \Phi(\mathcal{G}^0)$, $e \in \Phi(\mathcal{G}^1)$.

Furthermore, if $\phi(q_{[A]}) = Q_{[A]} = I_{(H,B)}$, then $p_A \in I_{(H,B)}$ which follows $A \in H$ and $[A] = [\emptyset]$ by [13, Theorem 6.12]. So, we have $q_{[A]} \neq 0$ for every $[A] \neq [\emptyset]$. Similarly, if $\phi(t_e) = s_e + I_{(H,B)} = I_{(H,B)}$ for $e \in \Phi(\mathcal{G}^1)$, then

$$p_{r'(e)} + I_{(H,B)} = (s_e + I_{(H,B)})^* (s_e + I_{(H,B)}) = I_{(H,B)}$$

and $p_{r'(e)} \in I_{(H,B)}$, which implies $r'(e) \in H$ or $[r'(e)] = [\emptyset]$. Thus if $e \in \Phi(\mathcal{G}^1)$, we have $t_e \neq 0$. Therefore, all generators of $C^*(\mathcal{G}/(H, B))$ are nonzero. \square

As for ultragraph C^* -algebras, one may easily show that

$$C^*(\mathcal{G}/(H, B)) = \overline{\text{span}} \{t_\alpha q_{[A]} t_\beta^* : \alpha, \beta \in (\mathcal{G}/(H, B))^*, r(\alpha) \cap [A] \cap r(\beta) \neq [\emptyset]\}.$$

4. UNIQUENESS THEOREMS

After defining quotient ultragraphs and associated C^* -algebras, in this section, we prove the gauge invariant and the Cuntz-Krieger uniqueness theorems for them. For this, we approach every quotient ultragraph C^* -algebra by graph C^* -algebras and then the corresponding uniqueness theorems for graph C^* -algebras will be applied. This approach is similar to that in [15, Section 2] and [17, Section 5] with more complications duo to the existence of singular vertices in quotient ultragraphs.

Again we fix an ultragraph \mathcal{G} , an admissible pair (H, B) in \mathcal{G} , and the quotient ultragraph $\mathcal{G}/(H, B) = (\Phi(G^0), \Phi(\mathcal{G}^0), \Phi(\mathcal{G}^1), r, s)$.

Definition 4.1. An element $[v] \in \Phi(G^0)$ is called a *sink* if $s^{-1}([v]) = \emptyset$. If $[v]$ emits finitely many edges of $\Phi(\mathcal{G}^1)$, $[v]$ is called a *regular vertex*. We say $[v]$ is a *singular vertex* if it is not regular. The set of all singular vertices of $\Phi(G^0)$ will be denoted by

$$\Phi_{\text{sg}}(G^0) := \{[v] \in \Phi(G^0) : [v] \text{ either is a sink or emits infinitely many edges of } \Phi(\mathcal{G}^1)\}.$$

Let $F \subseteq \Phi_{\text{sg}}(G^0) \cup \Phi(\mathcal{G}^1)$ be a finite subset and denote $F^0 := F \cap \Phi_{\text{sg}}(G^0)$ and $F^1 := F \cap \Phi(\mathcal{G}^1) = \{e_1, \dots, e_n\}$. For every $\omega = (\omega_1, \dots, \omega_n) \in \{0, 1\}^n \setminus \{0^n\}$, define

$$r(\omega) := \bigcap_{\omega_i=1} r(e_i) \setminus \bigcup_{\omega_j=0} r(e_j) \text{ and } R(\omega) := r(\omega) \setminus \bigcup F^0$$

and set

$\Gamma_0 := \{\omega \in \{0, 1\}^n \setminus \{0^n\} : \text{there are vertices } [v_1], \dots, [v_m] \text{ such that } R(\omega) = \bigcup_{i=1}^m [v_i] \text{ and } \emptyset \neq s^{-1}([v_i]) \subseteq F^1 \text{ for } 1 \leq i \leq m\},$

and

$$\Gamma := \{\omega \in \{0, 1\}^n \setminus \{0^n\} : R(\omega) \neq [\emptyset] \text{ and } \omega \notin \Gamma_0\}.$$

Note that for every two distinct $\omega, \nu \in \{0, 1\}^n \setminus \{0^n\}$, we have $r(\omega) \cap r(\nu) = [\emptyset]$. Now define the finite graph $G_F = (G_F^0, G_F^1, r_F, s_F)$ where

$$\begin{aligned} G_F^0 &:= F^0 \cup F^1 \cup \Gamma \\ G_F^1 &:= \{(e, f) \in F^1 \times F^1 : s(f) \subseteq r(e)\} \\ &\quad \cup \{(e, [v]) \in F^1 \times F^0 : [v] \subseteq r(e)\} \\ &\quad \cup \{(e, \omega) \in F^1 \times \Gamma : \omega_i = 1 \text{ when } e = e_i\} \end{aligned}$$

with $s_F(e, f) = s_F(e, [v]) = s_F(e, \omega) = e$ and $r_F(e, f) = f$, $r_F(e, [v]) = [v]$, $r_F(e, \omega) = \omega$.

Proposition 4.2. *Let $\mathcal{G}/(H, B)$ be a quotient ultragraph and $F \subseteq \Phi_{\text{sg}}(G^0) \cup \Phi(\mathcal{G}^1)$ be a finite subset. If $C^*(\mathcal{G}/(H, B)) = C^*(t_e, q_{[A]})$, then*

$$\begin{aligned} Q_e &:= t_e t_e^*, & Q_{[v]} &:= q_{[v]}(1 - \sum_{e \in F^1} t_e t_e^*), & Q_\omega &:= q_{R(\omega)}(1 - \sum_{e \in F^1} t_e t_e^*) \\ T_{(e, f)} &:= t_e Q_f, & T_{(e, [v])} &:= t_e Q_{[v]}, & T_{(e, \omega)} &:= t_e Q_\omega \end{aligned}$$

forms a Cuntz-Krieger G_F -family that generates the C^ -subalgebra $C^*(t_e, q_{[v]} : [v] \in F^0, e \in F^1)$ of $C^*(\mathcal{G}/(H, B))$. Moreover, every projection Q_* is nonzero.*

Proof. First, we see that all the projections Q_e , $Q_{[v]}$, and Q_ω are nonzero. Indeed, since every $[v] \in F^0$ is a singular vertex in $\mathcal{G}/(H, B)$, all $Q_{[v]}$'s are nonzero. Also, by definition, for every $\omega \in \Gamma$ we have $\omega \notin \Gamma_0$ and $R(\omega) \neq [\emptyset]$. If there is an edge $f \in \Phi(\mathcal{G}^1) \setminus F^1$ with $s(f) \subseteq R(\omega)$, then $0 \neq t_f t_f^* \leq Q_\omega$. If there is a sink $[w]$ such that $[w] \subseteq R(\omega) = r(\omega) \setminus \bigcup F^0$, then $0 \neq q_{[w]} \leq q_{R(\omega)}(1 - \sum_{e \in F^1} t_e t_e^*) = Q_\omega$. Thus, all Q_ω 's are nonzero. In addition, the projections Q_e , $Q_{[v]}$, and Q_ω are mutually orthogonal because of the factor $1 - \sum_{e \in F^1} t_e t_e^*$ and the definition of $R(\omega)$.

Now we show that $\{T_*, Q_*\}$ satisfies the relations of Remark 2.4 associated to the graph G_F .

(1): We have

$$\begin{aligned} T_{(e, f)}^* T_{(e, f)} &= Q_f t_e^* t_e Q_f = t_f t_f^* q_{r(e)} t_f t_f^* = t_f q_{r(f)} t_f^* = Q_f, \\ T_{(e, [v])}^* T_{(e, [v])} &= Q_{[v]} t_e^* t_e Q_{[v]} = Q_{[v]} q_{r(e)} Q_{[v]} = Q_{[v]}, \end{aligned}$$

and

$$T_{(e, \omega)}^* T_{(e, \omega)} = Q_\omega t_e^* t_e Q_\omega = Q_\omega q_{r(e)} Q_\omega = Q_\omega$$

because $Q_{[v]}, Q_\omega \leq q_{r(e)}$ whenever $(e, [v]), (e, \omega) \in G_F^1$.

(2): This relation may be verified as (1).

(3): Note that any vertex of $F^0 \cup \Gamma$ is a sink in G_F . So, for this relation fix some $e_i \in F^1$. If $q_{F^0} := \sum_{[v] \in F^0} q_{[v]}$, we first note that

(i)

$$q_{r(e_i)} \sum_{\substack{f \in F^1 \\ s(f) \subseteq r(e_i)}} Q_f = q_{r(e_i)} \sum_{\substack{f \in F^1 \\ s(f) \subseteq r(e_i)}} t_f t_f^* = q_{r(e_i)} \sum_{f \in F^1} t_f t_f^*;$$

(ii)

$$\begin{aligned} q_{r(e_i)} \sum_{\substack{[v] \in F^0, \\ [v] \subseteq r(e_i)}} Q_{[v]} &= q_{r(e_i)} \sum_{[v] \in F^0} q_{[v]} (1 - \sum_{e \in F^1} t_e t_e^*) \\ &= q_{r(e_i)} q_{F^0} (1 - \sum_{e \in F^1} t_e t_e^*); \end{aligned}$$

(iii)

$$\sum_{\omega \in \Gamma, \omega_i=1} Q_\omega = \sum_{\omega \in \Gamma, \omega_i=1} q_{R(\omega)} (1 - \sum_{e \in F^1} t_e t_e^*) = \sum_{\omega_i=1} q_{R(\omega)} (1 - \sum_{e \in F^1} t_e t_e^*),$$

$$\text{because } \sum_{\omega_i=1} q_{R(\omega)} = q_{r(e_i)} (1 - q_{F^0}).$$

Hence, by (i)-(iii), we get that

$$\begin{aligned} &\sum_{s(f) \subseteq r(e_i)} T_{(e_i, f)} + \sum_{[v] \in F^0, [v] \subseteq r(e_i)} T_{(e_i, [v])} + \sum_{\omega \in \Gamma, \omega_i=1} T_{(e_i, \omega)} \\ &= t_{e_i} \left(q_{r(e_i)} \sum_{e \in F^1} t_e t_e^* + q_{r(e_i)} q_{F^0} \left(\sum_{e \in F^1} t_e t_e^* \right) + q_{r(e_i)} (1 - q_{F^0}) \left(\sum_{e \in F^1} t_e t_e^* \right) \right) \\ &= t_{e_i} q_{r(e_i)} \left(\sum_{e \in F^1} t_e t_e^* + (q_{F^0} + 1 - q_{F^0}) (1 - \sum_{e \in F^1} t_e t_e^*) \right) \\ &= t_{e_i}. \end{aligned} \tag{4.1}$$

Now if e_i is not a sink as a vertex in G_F , i.e. $|\{x \in G_F^1 : s_F(x) = e_i\}| > 0$, we have

$$\begin{aligned} &\sum_{f \in F^1, s(f) \subseteq r(e_i)} T_{(e_i, f)} T_{(e_i, f)}^* + \sum_{[v] \in F^0, [v] \subseteq r(e_i)} T_{(e_i, [v])} T_{(e_i, [v])}^* + \sum_{\omega \in \Gamma, \omega_i=1} T_{(e_i, \omega)} T_{(e_i, \omega)}^* \\ &= \sum t_{e_i} Q_f t_{e_i}^* + \sum t_{e_i} Q_{[v]} t_{e_i}^* + \sum t_{e_i} Q_\omega t_{e_i}^* \\ &= t_{e_i} q_{r(e_i)} (\sum Q_f + \sum Q_{[v]} + \sum Q_\omega) t_{e_i}^* \\ &= t_{e_i} t_{e_i}^* = Q_{e_i}. \end{aligned}$$

Since the other vertices of G_F (the elements of $F^0 \cup \Gamma$) are sinks in G_F , we see that the relation (CK3) is satisfied.

Furthermore, the equation (4.1) shows that $t_{e_i} \in C^*(T_*, Q_*)$ for every $e_i \in F^1$. Also, if $[v] \in F^0$, then we have

$$\begin{aligned} Q_{[v]} + \sum_{e \in F^1, s(e)=[v]} Q_e &= t_{[v]}(1 - \sum_{e \in F^1} t_e t_e^*) + \sum_{e \in F^1, s(e)=[v]} t_e t_e^* \\ &= t_{[v]} - t_{[v]} \sum_{e \in F^1} t_e t_e^* + t_{[v]} \sum_{e \in F^1} t_e t_e^* \\ &= t_{[v]}. \end{aligned}$$

Consequently, the elements T_* 's and Q_* 's generate $C^*(\{t_e, q_{[v]} : e \in F^1, [v] \in F^0\})$. \square

Corollary 4.3. *If F is a finite subset of $\Phi_{\text{sg}}(G^0) \cup \Phi(\mathcal{G}^1)$, then $C^*(G_F)$ is isometrically isomorphic to the C^* -subalgebra of $C^*(\mathcal{G}/(H, B))$ generated by $\{t_e, q_{[v]} : [v] \in F^0, e \in F^1\}$.*

Proof. Suppose that X is the C^* -subalgebra generated by $\{t_e, q_{[v]} : [v] \in F^0, e \in F^1\}$ and let $\{T_x, Q_a : a \in G_F^0, x \in G_F^1\}$ be the Cuntz-Krieger G_F -family in Proposition 4.2. If $C^*(G_F) = C^*(s_x, p_a)$, then there exists a $*$ -homomorphism $\phi : C^*(G_F) \rightarrow X$ with $\phi(p_a) = Q_a$ and $\phi(s_x) = T_x$ for every $a \in G_F^0, x \in G_F^1$. Since all Q_a 's are nonzero by Proposition 4.2, the gauge invariant uniqueness theorem implies that ϕ is injective. Moreover, the family $\{T_x, Q_a\}$ generates all the C^* -subalgebra X , and so, ϕ is an isomorphism. \square

Note that if $F_1 \subseteq F_2$ are two finite subsets of $\Phi_{\text{sg}}(G^0) \cup \Phi(\mathcal{G}^1)$ and X_1, X_2 are the C^* -subalgebras of $C^*(\mathcal{G}/(H, B))$ corresponding to G_{F_1} and G_{F_2} , respectively, then we have $X_1 \subseteq X_2$ by Proposition 4.2.

Remark 4.4. By the relations of Definition 3.8, each $q_{[A]}, [A] \in \Phi(G^0)$, may be produced by the elements of

$$\{q_{[v]} : [v] \in \Phi_{\text{sg}}(G^0)\} \cup \{t_e : e \in \Phi(\mathcal{G}^1)\}$$

with finite operations. So, the $*$ -subalgebra of $C^*(\mathcal{G}/(H, B))$ generated by

$$\{q_{[v]} : [v] \in \Phi_{\text{sg}}(G^0)\} \cup \{t_e : e \in \Phi(\mathcal{G}^1)\}$$

is dense in $C^*(\mathcal{G}/(H, B))$.

As for ultragraph C^* -algebras, the universal property implies existence of the strongly continuous *gauge action* $\gamma : \mathbb{T} \rightarrow \text{Aut}(C^*(\mathcal{G}/(H, B)))$ such that

$$\gamma_z(t_e) = z t_e \text{ and } \gamma_z(q_{[A]}) = q_{[A]}$$

for every $[A] \in \Phi(G^0)$, $e \in \Phi(\mathcal{G}^1)$, and $z \in \mathbb{T}$.

Theorem 4.5 (The Gauge Invariant Uniqueness Theorem). *Let $\mathcal{G}/(H, B)$ be a quotient ultragraph and $\{T_e, Q_{[A]}\}$ be a representation for $\mathcal{G}/(H, B)$ such that $Q_{[A]} \neq 0$ for $[A] \neq [\emptyset]$. If $\pi_{T,Q} : C^*(\mathcal{G}/(H, B)) \rightarrow C^*(T_e, Q_{[A]})$ is the $*$ -homomorphism satisfying $\pi_{T,Q}(t_e) = T_e$, $\pi_{T,Q}(q_{[A]}) = Q_{[A]}$, and there is*

a strongly continuous action β of \mathbb{T} on $C^*(T_e, Q_{[A]})$ such that $\beta_z \circ \pi_{T,Q} = \pi_{T,Q} \circ \gamma_z$ for every $z \in \mathbb{T}$, then $\pi_{T,Q}$ is faithful.

Proof. Select an increasing sequence $\{F_n\}$ of finite subsets of $\Phi_{\text{sg}}(G^0) \cup \Phi(\mathcal{G}^1)$ such that $\bigcup_{n=1}^{\infty} F_n = \Phi_{\text{sg}}(G^0) \cup \Phi(\mathcal{G}^1)$. By Corollary 4.3, there are the isomorphisms

$$\pi_n : C^*(G_{F_n}) \rightarrow C^*(\{t_e, q_{[v]} : [v] \in F^0, e \in F^1\})$$

that respect the generators. Applying the gauge invariant uniqueness theorem for graph C^* -algebras shows that all $*$ -homomorphisms

$$\pi_{T,Q} \circ \pi_n : C^*(G_{F_n}) \rightarrow C^*(T_e, Q_{[A]})$$

are faithful. Hence, for each F_n , the restriction of $\pi_{T,Q}$ on the $*$ -subalgebra of $C^*(\mathcal{G}/(H, B))$ generated by $\{t_e, q_{[v]} : [v] \in F^0, e \in F^1\}$ is faithful, and so $\pi_{T,Q}$ is on the $*$ -subalgebra generated by $\{t_e, q_{[v]} : [v] \in \Phi_{\text{sg}}(G^0), e \in \Phi(\mathcal{G}^1)\}$. Since, the later subalgebra is dense in $C^*(\mathcal{G}/(H, B))$, we conclude that $\pi_{T,Q}$ is faithful. \square

For giving a Cuntz-Krieger uniqueness theorem for quotient ultragraphs, we need to extend condition (L) for them.

Definition 4.6. We say $\mathcal{G}/(H, B)$ satisfies *Condition (L)* if for every loop $\alpha = e_1 \dots e_n$ in $\mathcal{G}/(H, B)$, at least one of the following holds:

- (i) α has exits; which is, there exists $f \in \Phi(\mathcal{G}^1)$ with $s(f) \subseteq r(e_i)$ and $f \neq e_{i+1}$ for some $1 \leq i \leq n$, or
- (ii) $r(e_i) \neq s(e_{i+1})$ for some $1 \leq i \leq n$, where $e_{i+1} := e_1$.

Note that $r(e_i) \neq s(e_{i+1})$ in above means $r(e_i) \setminus s(e_{i+1}) \neq \emptyset$ because $s(e_{i+1}) \subseteq r(e_i)$. Also, since any element of $F^0 \cup \Gamma$ is a sink, every loop in a graph G_F is of the form $\tilde{\alpha} = (e_1, e_2) \dots (e_n, e_1)$ with $s(e_{i+1}) \subseteq r(e_i)$, $1 \leq i \leq n$, which is corresponding with the loop $\alpha = e_1 \dots e_n$ in $\mathcal{G}/(H, B)$.

Lemma 4.7. Let $F \subseteq \Phi_{\text{sg}}(G^0) \cup \Phi(\mathcal{G}^1)$ be a finite subset. If $\mathcal{G}/(H, B)$ satisfies *Condition (L)*, so does G_F .

Proof. Suppose that $\mathcal{G}/(H, B)$ satisfies *Condition (L)* and $\tilde{\alpha} = (e_1, e_2) \dots (e_n, e_1)$ is a loop in G_F . For the loop $\alpha = e_1 \dots e_n$ in $\mathcal{G}/(H, B)$, then

- (i) there exists $f \in \Phi(\mathcal{G}^1)$ with $s(f) \subseteq r(e_i)$ and $f \neq e_{i+1}$ for some $1 \leq i \leq n$, or
- (ii) $r(e_i) \neq s(e_{i+1})$ for some $1 \leq i \leq n$, where $e_{i+1} := e_1$.

In the case (i) if $f \in F^1$, (e_i, f) is an exit for $\tilde{\alpha}$. If $f \notin F^1$, for $[v] = s(f)$ we have either $[v] \notin F^0$ or

$$\exists \omega \in \Gamma \text{ with } \omega_i = 1 \text{ such that } [v] \subseteq R(\omega).$$

Thus $(e_i, [v])$ or (e_i, ω) is an exit for $\tilde{\alpha}$, respectively.

In the case (ii) suppose that $s(e_{i+1}) \subsetneq r(e_i)$ and $r(e_i)$ emits only the edge e_{i+1} of $\Phi(\mathcal{G}^1)$. So, there exists either $[v] \in F^0$ with $s(e_{i+1}) \neq [v] \subseteq r(e_i)$, or $\omega \in \Gamma$ with $\omega_i = 1$. Hence, $(e_i, [v])$ or (e_i, ω) is an exit for $\tilde{\alpha}$ in G_F , respectively. \square

Theorem 4.8 (The Cuntz-Krieger Uniqueness Theorem). *Suppose that $\mathcal{G}/(H, B)$ is a quotient ultragraph satisfying Condition (L). If $\{T_e, Q_A\}$ is a Cuntz-Krieger representation for $\mathcal{G}/(H, B)$ in which all the projection $Q_{[A]}$ are nonzero for $[A] \neq [\emptyset]$, then the $*$ -homomorphism $\pi_{T,Q} : C^*(\mathcal{G}/(H, B)) \rightarrow C^*(T_e, Q_{[A]})$ with $\pi_{T,Q}(t_e) = T_e$ and $\pi_{T,Q}(q_{[A]}) = Q_{[A]}$ is an isometrically isomorphism.*

Proof. It suffices to show that $\pi_{T,Q}$ is faithful. For this, let $\{F_n\}$ be an increasing sequence of finite subsets of $\Phi_{\text{sg}}(G^0) \cup \Phi(\mathcal{G}^1)$ such that $\bigcup_{n=1}^{\infty} F_n = \Phi_{\text{sg}}(G^0) \cup \Phi(\mathcal{G}^1)$. By Corollary 4.3, there are isomorphisms $\pi_n : C^*(G_{F_n}) \rightarrow C^*(\{t_e, q_{[v]} : [v] \in F_n^0, e \in F_n^1\})$ that respect the generators. Since all the graphs G_{F_n} satisfy Condition (L) by Lemma 4.7, the Cuntz-Krieger uniqueness theorem for graph C^* -algebras implies that the $*$ -homomorphisms

$$\pi_{T,Q} \circ \pi_n : C^*(G_{F_n}) \rightarrow C^*(T_e, Q_{[A]})$$

are faithful. Thus $\pi_{T,Q}$ is faithful on the $*$ -subalgebra of $C^*(\mathcal{G}/(H, B))$ generated by $\{t_e, q_{[v]} : [v] \in \Phi_{\text{sg}}(G^0), e \in \Phi(\mathcal{G}^1)\}$. Since this $*$ -subalgebra is dense in $C^*(\mathcal{G}/(H, B))$, we conclude that $\pi_{T,Q}$ is faithful on all $C^*(\mathcal{G}/(H, B))$. \square

5. GAUGE INVARIANT IDEALS

In this part, we consider gauge invariant ideals in a quotient ultragraph C^* -algebra $C^*(\mathcal{G}/(H, B))$. We may characterize the gauge invariant ideals of $C^*(\mathcal{G}/(H, B))$ using the ideal structure of $C^*(\mathcal{G})$ in [13, Section 6] so that they are corresponding with specific gauge invariant ideals of $C^*(\mathcal{G})$.

First, we show that $C^*(\mathcal{G}/(H, B))$ is isomorphic to a quotient of $C^*(\mathcal{G})$ which help us to apply the quotient ultragraphs for quotients of ultragraph C^* -algebras.

Proposition 5.1. *Let \mathcal{G} be an ultragraph. If (H, B) is an admissible pair in \mathcal{G} , then $C^*(\mathcal{G}/(H, B)) \cong C^*(\mathcal{G})/I_{(H,B)}$.*

Proof. Consider $I_{(H,B)}$ as an ideal of $C^*(\overline{\mathcal{G}})$ by Proposition 3.2. Let $C^*(\overline{\mathcal{G}}) = C^*(s_e, p_A)$ and $C^*(\mathcal{G}/(H, B)) = C^*(t_e, q_{[A]})$. If we define

$$T_e := s_e + I_{(H,B)} \text{ and } Q_{[A]} := p_A + I_{(H,B)}$$

for $[A] \in \Phi(\mathcal{G}^0)$, $e \in \Phi(\mathcal{G}^1)$, then we see that $\{T_e, Q_{[A]}\}$ is a representation for $\mathcal{G}/(H, B)$ in $C^*(\overline{\mathcal{G}})/I_{(H,B)}$. So, there is the $*$ -homomorphism $\phi : C^*(\mathcal{G}/(H, B)) \rightarrow C^*(\mathcal{G})/I_{(H,B)}$ with $\phi(t_e) = T_e$ and $\phi(q_{[A]}) = Q_{[A]}$. Further, all $Q_{[A]}$, $[A] \neq [\emptyset]$, are nonzero because $p_A + I_{(H,B)} = I_{(H,B)}$ implies $A \in H$. An application of Theorem 4.5 yields that ϕ is faithful. On the other hand, the family $\{T_e, Q_{[A]} : [A] \in \Phi(\mathcal{G}^0), e \in \Phi(\mathcal{G}^1)\}$ generates all the quotient $C^*(\mathcal{G})/I_{(H,B)}$, and so, ϕ is surjective. Therefore, ϕ is an isomorphism and the result follows. \square

Now fix some quotient ultragraph $\mathcal{G}/(H, B)$ of \mathcal{G} and let us assume $C^*(\mathcal{G}) = C^*(s_e, p_A)$ and $C^*(\mathcal{G}/(H, B)) = C^*(t_e, q_{[A]})$. If we define the partial order

$$(K_1, S_1) \subseteq (K_2, S_2) \iff K_1 \subseteq K_2 \text{ and } S_1 \subseteq K_2 \cup S_2,$$

then [13, Theorem 6.12] implies that the map $(K, S) \mapsto I_{(K, S)}$ is an one-to-one order preserving correspondence between admissible pairs in \mathcal{G} and the gauge invariant ideals of $C^*(\mathcal{G})$. Moreover, for every gauge invariant ideal $I_{(K, S)}$ in $C^*(\mathcal{G})$, it yields that

$$\{A \in \mathcal{G}^0 : p_A \in I_{(K, S)}\} = K$$

and

$$\{w \in B_K : p_w^K \in I_{(K, S)}\} = S.$$

Lemma 5.2. *Let $\phi : C^*(\mathcal{G}) \rightarrow C^*(\mathcal{G}/(H, B))$ be the canonical surjection described in Proposition 5.1, and let (K, S) be an admissible pair in \mathcal{G} . Then the image of ideal $I_{(K, S)}$ of $C^*(\mathcal{G})$ under ϕ is the ideal $J_{[K, S]}$ of $C^*(\mathcal{G}/(H, B))$ generated by*

$$\begin{aligned} & \left\{ q_{[\bar{A}]} , q_{[w']} : A \in K, w \in S \cap (B_H \setminus B) \right\} \\ & \cup \left\{ q_{[w]} - \sum_{s_{\mathcal{G}}(e)=w, r_{\mathcal{G}}(e) \notin K} t_e t_e^* : w \in S \setminus (B_H \setminus B) \right\}. \end{aligned}$$

Proof. Using the isomorphisms in Propositions 3.2 and 5.1, it is clear that $\phi(p_A) = q_{[\bar{A}]}$ and $\phi(p_w^H) = q_{[w']}$ for every $A \in K$ and $w \in S \cap (B_H \setminus B)$. Also, if $w \in S \setminus (B_H \setminus B)$, we have

$$\phi(p_w - \sum_{s_{\mathcal{G}}(e)=w, r_{\mathcal{G}}(e) \notin K} s_e s_e^*) = q_{[w]} - \sum_{s_{\mathcal{G}}(e)=w, r_{\mathcal{G}}(e) \notin K} t_e t_e^*$$

and the result follows. \square

Theorem 5.3. *Let $\mathcal{G}/(H, B)$ be a quotient ultragraph of \mathcal{G} . Then*

- (1) *the map $(K, S) \mapsto J_{[K, S]}$ is a one-to-one order preserving correspondence between the admissible pairs (K, S) in \mathcal{G} with $H \subseteq K$, $B \subseteq K \cup S$ and the gauge invariant ideals of $C^*(\mathcal{G}/(H, B))$;*
- (2) *if $J_{[K, S]}$ is a gauge invariant ideal in $C^*(\mathcal{G}/(H, B))$, then*

$$\frac{C^*(\mathcal{G}/(H, B))}{J_{[K, S]}} \cong C^*(\mathcal{G}/(K, S)).$$

Proof. (1): Suppose that J is a gauge invariant ideal in $C^*(\mathcal{G}/(H, B))$. If $\phi : C^*(\mathcal{G}) \rightarrow C^*(\mathcal{G}/(H, B))$ is the canonical map of Lemma 5.2 and $I = \phi^{-1}(J)$, then I is a gauge invariant ideal of $C^*(\mathcal{G})$ with $I_{(H, B)} \subseteq I$. [13, Theorem 6.12] says that there exists an admissible pair (K, S) in \mathcal{G} such that $I = I_{(K, S)}$. In particular, we have $H \subseteq K$ and $B \subseteq K \cup S$. Now by Lemma 5.2 we have $\phi(I_{(K, S)}) = J_{[K, S]}$ and since $\phi(I_{(K, S)}) = J$ either, we conclude that $J = J_{[K, S]}$.

(2): Let $J_{[K,S]}$ be a gauge invariant ideal in $C^*(\mathcal{G}/(H, B))$. Since $\phi(I_{(K,S)}) = J_{[K,S]}$ by Lemma 5.2, Proposition 5.1 implies that

$$\frac{C^*(\mathcal{G}/(H, B))}{J_{[K,S]}} \cong \frac{C^*(\mathcal{G})/I_{(H,B)}}{I_{(K,S)}/I_{(H,B)}} \cong \frac{C^*(\mathcal{G})}{I_{(K,S)}} \cong C^*(\mathcal{G}/(K, S)).$$

□

6. CONDITION (K)

We here consider Condition (K) for quotient ultragraphs. In [13, Proposition 7.3], it is shown that an ultragraph \mathcal{G} satisfies Condition (K) if and only if all ideals of $C^*(\mathcal{G})$ are gauge invariant. In this section, we extend the definition of Condition (K) to quotient ultragraphs and prove this result for quotient ultragraphs. In particular, we may obtain [13, Proposition 7.3] by applying our quotient ultragraphs instead of topological graphs of [11]. We first show that a quotient ultragraph $\mathcal{G}/(H, B)$ satisfies Condition (K) if and only if every quotient ultragraph $\mathcal{G}/(K, S)$ with $H \subseteq K$ satisfies Condition (L). Then main results will be obtained.

Definition 6.1. A quotient ultragraph $\mathcal{G}/(H, B)$ of \mathcal{G} is called to satisfy *Condition (K)* if for every vertex $v \in G^0 \setminus H$, v is the base of no loops or there are at least two loops α, β in \mathcal{G} based at v such that each one is not a subpath of the other; (if α is a subpath of β we simply write $\alpha \subseteq \beta$, and the otherwise is denoted by $\alpha \not\subseteq \beta$).

Remark 6.2. If $\mathcal{G}/(H, B)$ is an ultragraph, the above definition for Condition (K) is equivalent to that in [13, Definition 7.1].

Suppose that $\mathcal{G}/(H, B)$ is a quotient ultragraph of \mathcal{G} and $\gamma = e_1 \dots e_n$ is a loop in $\mathcal{G}/(H, B)$ such that there is no loops α with $s(\alpha) = s(\gamma)$ and $\alpha \not\subseteq \gamma$, $\gamma \not\subseteq \alpha$. If $\gamma^0 := \{s_{\mathcal{G}}(e_1), \dots, s_{\mathcal{G}}(e_n)\}$, define

$$X := \{r_{\mathcal{G}}(\alpha) \setminus \gamma^0 : \alpha \in \mathcal{G}^*, |\alpha| \geq 1, s_{\mathcal{G}}(\alpha) \in \gamma^0\},$$

$$Y := \left\{ \bigcup_{i=1}^n A_i : A_1, \dots, A_n \in X \cup H, n \in \mathbb{N} \right\},$$

and set

$$K_0 := \{B \in \mathcal{G}^0 : B \subseteq A \text{ for some } A \in Y\}.$$

We construct an specific saturated hereditary subset of \mathcal{G}^0 containing H as follows: for any $n \in \mathbb{N}$ inductively define

$$\begin{aligned} S_n := & \{w \in G^0 : 0 < |s_{\mathcal{G}}^{-1}(w)| < \infty \text{ and } r_{\mathcal{G}}(s_{\mathcal{G}}^{-1}(w)) \subseteq K_{n-1}\} \\ & \cup \{w \in B : w \notin K_{n-1} \text{ and } r_{\mathcal{G}}(s_{\mathcal{G}}^{-1}(w)) \subseteq K_{n-1}\} \end{aligned}$$

and

$$K_n := \{A \cup F : A \in K_{n-1} \text{ and } F \subseteq S_n \text{ is a finite subset}\}.$$

Then we can easily show that the set

$$K = \bigcup_{n=0}^{\infty} K_n = \left\{ A \cup F : A \in K_0 \text{ and } F \subseteq \bigcup_{n=1}^{\infty} S_n \text{ is a finite subset} \right\}$$

is hereditary and saturated in \mathcal{G} .

Lemma 6.3. *Suppose that $\mathcal{G}/(H, B)$ is a quotient ultragraph of \mathcal{G} and $\gamma = e_1 \dots e_n$ is a loop in $\mathcal{G}/(H, B)$ such that there is no loops α with $s(\alpha) = s(\gamma)$ and $\alpha \not\subseteq \gamma$, $\gamma \not\subseteq \alpha$. If we construct the set K as above, then K is a saturated hereditary subset of \mathcal{G}^0 such that $H \subseteq K$ and $B \subseteq K \cup B_K$. Moreover, we have $A \cap \gamma^0 = \emptyset$ for every $A \in K$.*

Proof. First we show inductively that every K_n is a hereditary subset of \mathcal{G}^0 by checking the conditions of Definition 2.5. To verify condition (i) for K_0 , let us take $e \in \mathcal{G}^1$ with $s_{\mathcal{G}}(e) \in K_0$. Then $s_{\mathcal{G}}(e) \in H \cup X$. If $s_{\mathcal{G}}(e) \in H$, we have $r_{\mathcal{G}}(e) \in H \subseteq K_0$ by the hereditary property of H . If $s_{\mathcal{G}}(e) \in X$, there is $\alpha \in \mathcal{G}^*$ such that $s_{\mathcal{G}}(\alpha) \in \gamma^0$ and $s_{\mathcal{G}}(e) \in r_{\mathcal{G}}(\alpha) \setminus \gamma^0$. Hence, $s_{\mathcal{G}}(\alpha e) = s_{\mathcal{G}}(\alpha) \in \gamma^0$. Also, $r_{\mathcal{G}}(\alpha e) \cap \gamma^0 = \emptyset$ because the otherwise implies the existence of a path $\beta \in \mathcal{G}^*$ with $s_{\mathcal{G}}(\beta) = s_{\mathcal{G}}(\gamma)$ and $\beta \not\subseteq \gamma$, $\gamma \not\subseteq \beta$ contradicting the hypothesis. It follows that

$$r_{\mathcal{G}}(e) = r_{\mathcal{G}}(\alpha e) = r_{\mathcal{G}}(\alpha e) \setminus \gamma^0 \in X \subseteq K_0.$$

Therefore, K_0 satisfies condition (i) of Definition 2.5. One can easily verify conditions (ii) and (iii) for K_0 . Also, for every $w \in S_n$, the range of all edges emitted by w belong to K_{n-1} by definition. Thus, we can inductively check that every K_n is hereditary either, and so is $K = \bigcup_{n=1}^{\infty} K_n$. The saturation property of K is may shown similar to the proof of [18, Lemma 3.12] and it is omitted.

Further, it is clear that $H \subseteq K$ as the fact $H \subseteq Y$. To see $B \subseteq K \cup B_K$, take an arbitrary vertex $w \in B \setminus K$. Then w is an infinite emitter and since $w \notin \bigcup_{n=1}^{\infty} S_n$ we have $r_{\mathcal{G}}(s_{\mathcal{G}}^{-1}(w)) \not\subseteq \bigcup_{n=1}^{\infty} K_n = K$. Thus w emits some edges into $\mathcal{G}^0 \setminus K$ which implies $w \in B_K$ as desired.

It remains to show $A \cap \gamma^0 = \emptyset$ for every $A \in K$. For this, first note that $A \cap \gamma^0 = \emptyset$ for any $A \in K_0$ because this property holds for any $A \in H$ and any $A \in X$. We claim that $(\bigcup_{n=1}^{\infty} S_n) \cap \gamma^0 = \emptyset$. Indeed, if $v = s_{\mathcal{G}}(e_i) \in \gamma^0$ for some $e_i \in \gamma$, then $r_{\mathcal{G}}(e_i) \cap \gamma^0 \neq \emptyset$ and $r_{\mathcal{G}}(e_i) \notin K_0$. Hence, $\{r_{\mathcal{G}}(e) : e \in \mathcal{G}^1, s_{\mathcal{G}}(e) = v\} \not\subseteq K_0$ which yields $v \notin S_1$. So, we have $S_1 \cap \gamma^0 = \emptyset$. An inductive argument shows that $S_n \cap \gamma^0 = \emptyset$ for all $n \geq 1$, and the claim holds. Now since

$$K = \bigcup_{n=1}^{\infty} K_n = \{A \cup F : A \in K_0 \text{ and } F \subseteq \bigcup_{n=1}^{\infty} S_n \text{ is a finite subset}\},$$

we conclude that $A \cap \gamma^0 = \emptyset$ for all $A \in K$. \square

Proposition 6.4. *A quotient ultragraph $\mathcal{G}/(H, B)$ of \mathcal{G} satisfies Condition (K) if and only if for every admissible pair (K, S) in \mathcal{G} with $H \subseteq K$ and $B \subseteq K \cup S$, the quotient ultragraph $\mathcal{G}/(K, S)$ satisfies Condition (L).*

Proof. Suppose that $\mathcal{G}/(H, B)$ satisfies Condition (K) and (K, S) is an admissible pair in \mathcal{G} with $H \subseteq K$. Let $\alpha = e_1 \dots e_n$ be a loop in $\mathcal{G}/(K, S)$. Since α is also a loop in $\mathcal{G}/(H, B)$ and $\mathcal{G}/(H, B)$ satisfies Condition (K), there is a loop $\beta = f_1 \dots f_m$ in \mathcal{G} with $s_{\mathcal{G}}(\alpha) = s_{\mathcal{G}}(\beta)$, and neither $\alpha \subseteq \beta$ nor $\beta \subseteq \alpha$. Without loss of generality, assume that $e_1 \neq f_1$. By the fact $s_{\mathcal{G}}(\alpha) = s_{\mathcal{G}}(\beta) \in r_{\mathcal{G}}(\beta)$, we have $r_{\mathcal{G}}(\beta) \not\subseteq K$, and so $r_{\mathcal{G}}(f_1) \not\subseteq K$ by the hereditary property of K . Hence, f_1 is an exit for α in $\mathcal{G}/(K, S)$ which concludes that $\mathcal{G}/(K, S)$ satisfies Condition (L).

For the converse, suppose that $\mathcal{G}/(H, B)$ does not satisfy Condition (K) and let $\gamma = e_1 \dots e_n$ be a loop in \mathcal{G} such that there is no loops α in $\mathcal{G}/(H, B)$ with $s(\alpha) = s(\gamma)$, and $\alpha \not\subseteq \gamma$, $\gamma \not\subseteq \alpha$. As Lemma 6.3, construct the hereditary and saturated subset K of \mathcal{G}^0 and consider the quotient ultragraph $\mathcal{G}/(K, B_K) = (\Psi(G^0), \Psi(\mathcal{G}^0), \Psi(\mathcal{G}^1), r', s')$. Denote the equivalent classes in $\Psi(\mathcal{G}^0)$ by $[\cdot]$. We show that γ as a loop in $\mathcal{G}/(K, B_K)$ has no exits and $r'(e_i) = s'(e_{i+1})$ for $1 \leq i \leq n$. If f is an exit for γ in $\mathcal{G}/(K, B_K)$ with $s'(f) = s'(e_j)$ and $f \neq e_j$, then $r_{\mathcal{G}}(f) \not\subseteq K$ and $r_{\mathcal{G}}(f) \cap \gamma^0 \neq \emptyset$ (if $r_{\mathcal{G}}(f) \cap \gamma^0 = \emptyset$, then $r_{\mathcal{G}}(f) = r_{\mathcal{G}}(f) \setminus \gamma^0 \in X \subseteq K$, a contradiction). Let $s_{\mathcal{G}}(e_l) \in r_{\mathcal{G}}(f)$, for some $e_l \in \gamma$. If we set $\alpha := e_1 \dots e_{j-1} f e_l \dots e_n$, α is a loop in \mathcal{G} with $s_{\mathcal{G}}(\alpha) = s_{\mathcal{G}}(\gamma)$, and $\alpha \not\subseteq \gamma$, $\gamma \not\subseteq \alpha$, which contradicts the hypothesis. Therefore, γ has no exits in $\mathcal{G}/(K, B_K)$. Moreover, we have $r'(e_i) \cap [\gamma^0] = s'(e_{i+1})$ for each $1 \leq i \leq n$, because the otherwise gives an exit for γ in $\mathcal{G}/(K, B_K)$ by the construction of K . Hence,

$$r'(e_i) \setminus s'(e_{i+1}) = r'(e_i) \setminus [\gamma^0] = [\emptyset]$$

and so $r'(e_i) = s'(e_{i+1})$ (note that the fact $r_{\mathcal{G}}(e_i) \setminus \gamma^0 \in K$ implies $r'(e_i) \setminus [\gamma^0] = [r_{\mathcal{G}}(e_i) \setminus \gamma^0] = [\emptyset]$). Consequently, the quotient ultragraph $\mathcal{G}/(K, B_K)$ does not satisfy Condition (L). \square

As for the C^* -algebras of graphs and ultragraphs, if $\mathcal{G}/(H, B)$ does not satisfies Condition (L), then $C^*(\mathcal{G}/(H, B))$ contains ideals which are not gauge invariant.

Lemma 6.5. *Let $\mathcal{G}/(H, B) = (\Phi(G^0), \Phi(\mathcal{G}^0), \Phi(\mathcal{G}^1), r, s)$ be a quotient ultragraph of \mathcal{G} . If $\mathcal{G}/(H, B)$ does not satisfy Condition (L), then $C^*(\mathcal{G}/(H, B))$ contains an ideal Morita equivalent to $C(\mathbb{T})$. In particular, $C^*(\mathcal{G}/(H, B))$ contains non-gauge invariant ideals.*

Proof. Suppose that $\gamma = e_1 \dots e_n$ is a loop in $\mathcal{G}/(H, B)$ without exits and $r(e_i) = s(e_{i+1})$ for $1 \leq i \leq n$. If $C^*(\mathcal{G}/(H, B)) = C^*(t_e, q_{[A]})$, for each i we have

$$t_{e_i}^* t_{e_i} = q_{r(e_i)} = q_{s(e_{i+1})} = t_{e_{i+1}} t_{e_{i+1}}^*.$$

Write $[v] := s(\gamma)$ and let I_{γ} be the ideal of $C^*(\mathcal{G}/(H, B))$ generated by $q_{[v]}$. Since γ has no exits in $\mathcal{G}/(H, B)$ and

$$q_{s(e_i)} = (t_{e_i} \dots t_{e_n}) q_{[v]} (t_{e_n}^* \dots t_{e_i}^*),$$

for every $1 \leq i \leq n$, an easy argument shows that

$$I_\gamma = \overline{\text{span}} \{t_\alpha q_{[v]} t_\beta^* : \alpha, \beta \in (\mathcal{G}/(H, B))^*, [v] \subseteq r(\alpha) \cap r(\beta)\}.$$

So, we have

$$q_{[v]} I_\gamma q_{[v]} = \overline{\text{span}} \{(t_\gamma)^n q_{[v]} (t_\gamma^*)^m : m, n \geq 0\},$$

where $(t_\gamma)^0 = (t_\gamma^*)^0 = q_{[v]}$. We show that $q_{[v]} I_\gamma q_{[v]}$ is a full corner in I_γ which is isometrically isomorphic to $C(\mathbb{T})$. For this, associated to the graph E

$$f \begin{array}{c} \curvearrowright \\ \end{array} w$$

define $Q_w := q_{[v]}$ and $T_f := t_\gamma (= t_\gamma q_{[v]})$. Then $\{T_f, Q_w\}$ is a Cuntz-Krieger E -family in $q_{[v]} I_\gamma q_{[v]}$. Assume $C^*(E) = C^*(s_f, p_w)$. Since $Q_w \neq 0$, the gauge-invariant uniqueness theorem for graph C^* -algebras implies that the $*$ -homomorphism $\phi : C^*(E) \rightarrow q_{[v]} I_\gamma q_{[v]}$ with $p_w \mapsto Q_w$ and $s_f \mapsto T_f$ is faithful. Also, the C^* -algebra $q_{[v]} I_\gamma q_{[v]}$ is generated by $\{T_f, Q_w\}$, and hence, ϕ is an isomorphism. Since $C^*(E) \cong C(\mathbb{T})$, it follows that $q_{[v]} I_\gamma q_{[v]}$ is isometrically isomorphic to $C(\mathbb{T})$. The fullness of $q_{[v]} I_\gamma q_{[v]}$ in I_γ may be easily shown. Indeed, if J is an ideal in I_γ with $q_{[v]} I_\gamma q_{[v]} \subseteq J$, we see that $q_{[v]} \in J$. Since J is also an ideal in $C^*(\mathcal{G}/(H, B))$, we must have $J = I_\gamma$. Therefore, I_γ is Morita equivalent to $C(\mathbb{T})$ as desired.

On the other hand, $C(\mathbb{T})$ has infinitely many non-gauge invariant ideals (corresponding to closed subsets $\emptyset \neq U \subsetneq \mathbb{T}$). Therefore, the corner $q_{[v]} I_\gamma q_{[v]}$ contains non-gauge invariant ideals, and so does I_γ by the Morita-equivalence. It follows the second statement of lemma because every ideal of I_γ is also an ideal of $C^*(\mathcal{G}/(H, B))$. \square

Now we can prove the main result of this section that is a generalization of [1, Corollary 3.8] and [13, Proposition 7.3].

Theorem 6.6. *A quotient ultragraph $\mathcal{G}/(H, B)$ of \mathcal{G} satisfies Condition (K) if and only if all ideals of $C^*(\mathcal{G}/(H, B))$ are gauge invariant.*

Proof. Suppose that $\mathcal{G}/(H, B)$ satisfies Condition (K). Take an arbitrary ideal J in $C^*(\mathcal{G}/(H, B))$ and let I be its corresponding ideal in $C^*(\mathcal{G})$ with $I_{(H, B)} \subseteq I$ by Proposition 5.1. If $C^*(\mathcal{G}) = C^*(s_e, p_A)$ and set

$$K := \{A \in \mathcal{G}^0 : p_A \in I\}, \quad S := \left\{ w \in B_K : p_w - \sum_{s(e)=w, r(e) \notin K} s_e s_e^* \in I \right\},$$

then [18, Lemma 3.4] implies that (K, S) is an admissible pair in \mathcal{G} . Since $I_{(K, S)} \subseteq I$, the map

$$\phi : \frac{C^*(\mathcal{G})}{I_{(K, S)}} \rightarrow \frac{C^*(\mathcal{G})}{I}$$

with $\phi(a + I_{(K,S)}) = a + I$, for every $a \in C^*(\mathcal{G})/I_{(K,S)}$, is a well-defined $*$ -homomorphism. Let us denote $\pi : C^*(\mathcal{G}/(K, S)) \rightarrow C^*(\mathcal{G})/I_{(K,S)}$ the isomorphism of Proposition 5.1. Since the quotient ultragraph $\mathcal{G}/(K, S)$ satisfies Condition (L) by Proposition 6.4, the Cuntz-Krieger uniqueness theorem, Theorem 4.8, implies that $\phi \circ \pi$ is injective. So, ϕ is also injective which yields $I = I_{(K,S)}$ and I is gauge invariant. Now as J is the image of $I_{(K,S)}$ into the quotient $C^*(\mathcal{G})/I_{(H,B)} \cong C^*(\mathcal{G}/(H, B))$, we conclude that $J = J_{[K,S]}$ that is gauge invariant in $C^*(\mathcal{G}/(H, B))$.

Conversely, assume that $\mathcal{G}/(H, B)$ does not satisfy Condition (K). By Proposition 6.4, there exists an admissible pair (K, S) in \mathcal{G} with $H \subseteq K$ and $B \subseteq K \cup S$ such that the quotient ultragraph $\mathcal{G}/(K, S)$ does not satisfy Condition (L). Note that we have $I_{(H,B)} \subseteq I_{(K,S)}$ and if $J_{[K,S]}$ is the image of $I_{(K,S)}$ into $C^*(\mathcal{G})/I_{(H,B)} \cong C^*(\mathcal{G}/(H, B))$, then $C^*(\mathcal{G}/(H, B))/J_{[K,S]} \cong C^*(\mathcal{G}/(K, S))$. Lemma 6.5 says that $C^*(\mathcal{G}/(K, S))$ contains infinitely many ideals which are not gauge invariant. On the other hand, for every gauge invariant ideal J of $C^*(\mathcal{G}/(H, B))$, the ideal $J + J_{[K,S]}$ is also gauge invariant in $C^*(\mathcal{G}/(H, B))/J_{[K,S]} \cong C^*(\mathcal{G}/(K, S))$. This concludes that $C^*(\mathcal{G}/(H, B))$ contains non-gauge invariant ideals either. \square

We gather all the results of this section in the following corollary. Recall from [3] that the *real rank* of a unital C^* -algebra A is the smallest integer $\text{RR}(A)$ such that for every $\varepsilon > 0$, each $n \leq \text{RR}(A) + 1$ and each n -tuple (x_1, \dots, x_n) of self-adjoint elements in A , there is an n -tuple (y_1, \dots, y_n) of self-adjoint elements of A such that $\sum_{i=1}^n y_i^2$ is invertible and

$$\left\| \sum_{i=1}^n (x_i - y_i)^2 \right\| < \varepsilon.$$

If A is non-unital, $\text{RR}(A)$ is the real rank of its unitization.

Corollary 6.7. *Let H be a saturated hereditary subset of \mathcal{G}^0 . Then the following conditions are equivalent:*

- (1) *the quotient ultragraph $\mathcal{G}/(H, B_H)$ satisfies Condition (K);*
- (2) *the quotient ultragraph $\mathcal{G}/(H, B)$ satisfies Condition (K) for some $B \subseteq B_H$;*
- (3) *for every admissible pair (K, S) with $H \subseteq K$, $\mathcal{G}/(K, S)$ satisfies Condition (L);*
- (4) *if $B \subseteq B_H$, all ideals of $C^*(\mathcal{G}/(H, B))$ are gauge invariant;*
- (5) *if $B \subseteq B_H$, the real rank of $C^*(\mathcal{G}/(H, B))$ is zero.*

Proof. The equivalence of conditions (1)-(4) has been shown in this section. We show the implications (2) \Rightarrow (5) and (5) \Rightarrow (3). Suppose that $B \subseteq B_H$ and $\mathcal{G}/(H, B)$ satisfies Condition (K). Select an increasing sequence $\{F_n\}_{n=1}^\infty$ of finite subsets of $\Phi_{\text{sg}}(G^0) \cup \Phi(\mathcal{G}^1)$ such that $\cup_{n=1}^\infty F_n = \Phi_{\text{sg}}(G^0) \cup \Phi(\mathcal{G}^1)$. Then, similar to the proof of Theorem 4.5, $C^*(\mathcal{G}/(H, B))$ is isomorphic to the inductive limit $\varinjlim C^*(G_{F_n})$. Since loops of each G_{F_n} come from those of $\mathcal{G}/(H, B)$, we may select F_n 's such that any finite graph G_{F_n} satisfies

Condition (K), and so, the real rank of every C^* -algebra $C^*(G_{F_n})$ is zero [10, Theorem 4.1]. Thus $C^*(\mathcal{G}/(H, B))$ has real rank zero by [3, Proposition 3.1]. This proves the implication (2) \Rightarrow (5).

For (5) \Rightarrow (3), suppose that there is an admissible pair (K, S) with $H \subseteq K$ such that the quotient ultragraph $\mathcal{G}/(K, S)$ does not satisfy Condition (L). We may also assume that $B \subseteq K \cup S$ by Lemma 6.3. Then, by Lemma 6.5, $C^*(\mathcal{G}/(K, S))$ contains an ideal Morita equivalent to $C(\mathbb{T})$, and so $\text{RR}(C^*(\mathcal{G}/(K, S))) \neq 0$ by [3, Corollary 2.8]. As $C^*(\mathcal{G}/(K, S))$ is a quotient of $C^*(\mathcal{G}/(H, B))$, [3, Theorem 3.14] implies that $\text{RR}(C^*(\mathcal{G}/(H, B))) \neq 0$ either. \square

For ultragraph C^* -algebras, Corollary 6.7 implies both [13, Proposition 7.3] and [12, Proposition 5.26] because every ultragraph may be considered as a quotient ultragraph with trivial admissible pair (\emptyset, \emptyset) . However, our method is quite different from those of [13] and [12].

Corollary 6.8. *An ultragraph \mathcal{G} satisfies Condition (K) if and only if all ideals of $C^*(\mathcal{G})$ are gauge invariant if and only if the real rank of $C^*(\mathcal{G})$ is zero.*

7. PRIMITIVE IDEALS IN $C^*(\mathcal{G})$

In this section, we present an application of quotient ultragraphs to characterize primitive gauge invariant ideals of ultragraph C^* -algebras. Then we can describe all primitive gauge invariant ideals in a quotient ultragraph C^* -algebra $C^*(\mathcal{G}/(H, B))$ as well. Recall that since every ultragraph C^* -algebra $C^*(\mathcal{G})$ is separable (as \mathcal{G}^0 is assumed to be countable), an ideal of $C^*(\mathcal{G})$ is primitive if and only if it is prime [4, Corollaire 1].

Definition 7.1. Let \mathcal{G} be an ultragraph. For two sets $A, B \in \mathcal{G}^0$, we denote $A \geq B$ if either $B \subseteq A$ or there exists $\alpha \in \mathcal{G}^*$ with $|\alpha| \geq 1$ such that $s(\alpha) \in A$ and $B \subseteq r(\alpha)$. We simply write $A \geq v$, $v \geq B$, and $v \geq w$ if $A \geq \{v\}$, $\{v\} \geq B$, and $\{v\} \geq \{w\}$, respectively. A subset $M \subseteq \mathcal{G}^0$ is said to satisfy the *downward property* if for every $A, B \in M$, there exists $C \in M$ such that $A, B \geq C$.

To prove Proposition 7.3 below, we need a simple lemma.

Lemma 7.2. *If $\mathcal{G}/(H, B)$ satisfies Condition (L), then every nonzero ideal of $C^*(\mathcal{G}/(H, B))$ contains some projection $q_{[A]}$ with $[A] \neq [\emptyset]$.*

Proof. Take an arbitrary ideal J in $C^*(\mathcal{G}/(H, B))$. If there are no $q_{[A]} \in J$ with $[A] \neq [\emptyset]$, then the Cuntz-Krieger uniqueness theorem implies that the quotient homomorphism $\phi : C^*(\mathcal{G}/(H, B)) \rightarrow C^*(\mathcal{G}/(H, B))/J$ is injective. Hence, we have $J = \ker \phi = (0)$. \square

Proposition 7.3. *Let H be a saturated hereditary subset of \mathcal{G}^0 . Then the ideal $I_{(H, B_H)}$ in $C^*(\mathcal{G})$ is primitive if and only if the quotient ultragraph $\mathcal{G}/(H, B_H)$ satisfies Condition (L) and the collection $\mathcal{G}^0 \setminus H$ has the downward property.*

Proof. Suppose that $I_{(H, B_H)}$ is a primitive ideal in $C^*(\mathcal{G})$. Since $C^*(\mathcal{G})/I_{(H, B_H)} \cong C^*(\mathcal{G}/(H, B_H))$, the zero ideal of $C^*(\mathcal{G}/(H, B_H))$ is primitive. If $\mathcal{G}/(H, B_H)$ does not satisfy Condition (L), then $C^*(\mathcal{G}/(H, B_H))$ contains an ideal J Morita-equivalent to $C(\mathbb{T})$ by Lemma 6.5. Let I_1, I_2 be two nonzero ideals in $C(\mathbb{T})$ with $I_1 \cap I_2 = (0)$ and J_1, J_2 be their correspondent ideals in J . Since every ideal of J is also an ideal in $C^*(\mathcal{G}/(H, B_H))$, it follows that J_1, J_2 are two nonzero ideals of $C^*(\mathcal{G}/(H, B_H))$ with $J_1 \cap J_2 = (0)$, contradicting the primness of $C^*(\mathcal{G}/(H, B_H))$.

Now we show the downward property for $M := \mathcal{G}^0 \setminus H$. Take some $A, B \in M$ and consider the ideals

$$J_1 := C^*(\mathcal{G}/(H, B_H))q_{[A]}C^*(\mathcal{G}/(H, B_H))$$

and

$$J_2 := C^*(\mathcal{G}/(H, B_H))q_{[B]}C^*(\mathcal{G}/(H, B_H))$$

in $C^*(\mathcal{G}/(H, B_H))$ generated by $q_{[A]}$ and $q_{[B]}$, respectively. Since $A, B \notin H$, the projections $q_{[A]}, q_{[B]}$ are nonzero, and so are J_1, J_2 . The primness of $C^*(\mathcal{G}/(H, B_H))$ follows that the ideal

$$J_1 J_2 = C^*(\mathcal{G}/(H, B_H))q_{[A]}C^*(\mathcal{G}/(H, B_H))q_{[B]}C^*(\mathcal{G}/(H, B_H))$$

is nonzero and so $q_{[A]}C^*(\mathcal{G}/(H, B_H))q_{[B]} \neq \{0\}$. As the set

$$\text{span} \{t_\alpha q_{[D]} t_\beta^* : \alpha, \beta \in (\mathcal{G}/(H, B))^*, r(\alpha) \cap [D] \cap r(\beta) \neq [\emptyset]\}$$

is dense in $C^*(\mathcal{G}/(H, B_H))$, there exist $\alpha, \beta \in (\mathcal{G}/(H, B_H))^*$ and $[D] \in \Phi(\mathcal{G}^0)$ such that $q_{[A]} t_\alpha q_{[D]} t_\beta^* q_{[B]} \neq 0$. In this case we have $s(\alpha) \subseteq [A]$ and $s(\beta) \subseteq [B]$. Thus, for $C := r_{\mathcal{G}}(\alpha) \cap D \cap r_{\mathcal{G}}(\beta)$ we have $A, B \geq C$.

For the converse, assume that $\mathcal{G}/(H, B_H)$ satisfies Condition (L) and $M = \mathcal{G}^0 \setminus H$ has the downward property. Let J_1 and J_2 be two nonzero ideals of $C^*(\mathcal{G}/(H, B_H))$. By Lemma 7.2, there are nonzero projections $q_{[A]} \in J_1$ and $q_{[B]} \in J_2$. Since $A, B \notin H$, the downward property yields that there exists $C \in M$ such that $A, B \geq C$. Hence, $q_{[C]} \in J_1 \cap J_2$ and so $J_1 \cap J_2 \neq (0)$. It follows that $C^*(\mathcal{G}/(H, B_H))$ is primitive and consequently, $I_{(H, B_H)}$ is a primitive ideal in $C^*(\mathcal{G})$. \square

The next proposition indicates another kind of primitive ideals in $C^*(\mathcal{G})$.

Proposition 7.4. *Let (H, B) be an admissible pair in \mathcal{G} and $B = B_H \setminus \{w\}$. Then the ideal $I_{(H, B)}$ of $C^*(\mathcal{G})$ is primitive if and only if $A \geq w$ for all $A \in \mathcal{G}^0 \setminus H$.*

Proof. Suppose that $I_{(H, B)}$ is a primitive ideal and take an arbitrary $A \in \mathcal{G}^0 \setminus H$. If $\overline{A} := A \cup \{v' : v \in A \cap (B_H \setminus B)\}$, then $q_{\overline{A}}$ and $q_{[w']}$ are two nonzero projections in $C^*(\mathcal{G}/(H, B))$. Let $J_{\overline{A}}, J_{[w']}$ be two ideals of $C^*(\mathcal{G}/(H, B))$ respectively generated by $q_{\overline{A}}$ and $q_{[w']}$, which are

$$J_{\overline{A}} := C^*(\mathcal{G}/(H, B))q_{\overline{A}}C^*(\mathcal{G}/(H, B))$$

and

$$J_{[w']} := C^*(\mathcal{G}/(H, B))_{q_{[w']}} C^*(\mathcal{G}/(H, B)).$$

The primness of $C^*(\mathcal{G}/(H, B)) \cong C^*(\mathcal{G})/I_{H,B}$ implies that

$$J_{[\bar{A}]} J_{[w']} = C^*(\mathcal{G}/(H, B))_{q_{[\bar{A}]}} C^*(\mathcal{G}/(H, B))_{q_{[w']}} C^*(\mathcal{G}/(H, B)) \neq \{0\},$$

and so, $q_{[\bar{A}]} C^*(\mathcal{G}/(H, B))_{q_{[w']}} \neq \{0\}$. Thus there exist $\alpha, \beta \in (\mathcal{G}/(H, B))^*$ such that $q_{[\bar{A}]} t_\alpha t_\beta^* q_{[w']} \neq 0$. Since $[w']$ is a sink in $\mathcal{G}/(H, B)$, we have $q_{[\bar{A}]} t_\alpha q_{[w']} \neq 0$. If $|\alpha| = 0$, then $[w'] \subseteq [\bar{A}]$, $w' \in \bar{A}$ and $w \in A$. If $|\alpha| \geq 1$, we have $s(\alpha) \subseteq [\bar{A}]$ and $[w'] \subseteq r(\alpha)$, which yield $s_{\mathcal{G}}(\alpha) \in A$ and $w \in r_{\mathcal{G}}(\alpha)$. Therefore, in the both cases we have $A \geq w$.

Conversely, assume that $A \geq w$ for every $A \in \mathcal{G}^0 \setminus H$. Then $\mathcal{G}^0 \setminus H$ satisfies the downward property. Also, the hypothesis implies that for every $[A] \in \Phi(\mathcal{G}^0)$, there exists $\alpha \in (\mathcal{G}/(H, B))^*$ such that $s(\alpha) \subseteq [A]$ and $[w'] \subseteq r(\alpha)$. So, since $[w']$ is a sink in $\mathcal{G}/(H, B)$, we see that the quotient ultragraph $\mathcal{G}/(H, B)$ satisfies Condition (L). Now similar to the last part of the proof of Proposition 7.3, we can show that $I_{(H,B)}$ is a primitive ideal. \square

We could check Condition (L) for a quotient ultragraph $\mathcal{G}/(H, B)$ by verifying it in the initial ultragraph \mathcal{G} . For this, we give the following definition.

Definition 7.5. Let H be a saturated hereditary subset of \mathcal{G}^0 and denote $M := \mathcal{G}^0 \setminus H$. A loop $\alpha = e_1 \dots e_n$ is said to be in $\mathcal{G} \setminus H$ if $r_{\mathcal{G}}(\alpha) \in M$. Also, we say that α has an *exit* in $\mathcal{G} \setminus H$ if either $r_{\mathcal{G}}(e_i) \setminus s_{\mathcal{G}}(e_{i+1}) \in M$ for some i , or there is an edge f with $r_{\mathcal{G}}(f) \in M$ such that $s_{\mathcal{G}}(f) = s_{\mathcal{G}}(e_i)$ and $f \neq e_i$, for some $1 \leq i \leq n$.

It is easy to verify that a quotient ultragraph $\mathcal{G}/(H, B_H)$ satisfies Condition (L) if and only if every loop in $\mathcal{G} \setminus H$ has an exit in $\mathcal{G} \setminus H$. Now we characterize all primitive gauge invariant ideals of an ultragraph C^* -algebra $C^*(\mathcal{G})$ that is a generalization of [1, Theorem 4.7] and [5, Theorem 4.5].

Theorem 7.6. *Let \mathcal{G} be an ultragraph. A gauge invariant ideal $I_{(H,B)}$ of $C^*(\mathcal{G})$ is primitive if and only if one of the following holds:*

- (1) $B = B_H$, $\mathcal{G}^0 \setminus H$ satisfies the downward property, and every loop in $\mathcal{G} \setminus H$ has exits in $\mathcal{G} \setminus H$.
- (2) $B = B_H \setminus \{w\}$ for some $w \in B_H$, and $A \geq w$ for all $A \in \mathcal{G}^0 \setminus H$.

Proof. Suppose that $I_{(H,B)}$ is a primitive ideal. Then $C^*(\mathcal{G}/(H, B)) \cong C^*(\mathcal{G})/I_{(H,B)}$ is a primitive C^* -algebra. We claim that $|B_H \setminus B| \leq 1$. Indeed, if w_1, w_2 are two distinct vertices in $B_H \setminus B$, then $[w'_1]$ and $[w'_2]$ are two sinks in $\mathcal{G}/(H, B)$. The primitivity of $C^*(\mathcal{G}/(H, B))$ implies that the corner $q_{[w'_1]} C^*(\mathcal{G}/(H, B))_{q_{[w'_2]}}$ is nonzero, and so there exist $\alpha, \beta \in (\mathcal{G}/(H, B))^*$ such that $q_{[w'_1]} t_\alpha t_\beta^* q_{[w'_2]} \neq 0$. But we must have $|\alpha| = |\beta| = 0$ and hence, $q_{[w'_1]} q_{[w'_2]} \neq 0$ which is impossible because $q_{[w'_1]} q_{[w'_2]} = q_{[\{w'_1\} \cap \{w'_2\}]} = 0$. Thus, the claim holds. Now Propositions 7.3 and 7.4 yield the result. \square

Corollary 6.8 says that if \mathcal{G} satisfies Condition (K), every ideal of $C^*(\mathcal{G})$ is of the form $I_{(H,B)}$. So, we have the following.

Corollary 7.7. *If \mathcal{G} satisfies Condition (K), then Theorem 7.6 describes all primitive ideals of $C^*(\mathcal{G})$.*

Now using Theorem 7.6 we can determine primitive gauge invariant ideals of a quotient ultragraph C^* -algebra $C^*(\mathcal{G}/(H, B))$. Recall from Theorem 5.3 that every gauge invariant ideal of $C^*(\mathcal{G}/(H, B))$ is of the form $J_{[K,S]}$ with $H \subseteq K$ and $B \subseteq K \cup S$. Since

$$\frac{C^*(\mathcal{G}/(H, B))}{J_{[K,S]}} \cong C^*(\mathcal{G}/(K, S)) \cong \frac{C^*(\mathcal{G})}{I_{(K,S)}}$$

by Theorem 5.3, a gauge invariant ideal $J_{[K,S]}$ is primitive in $C^*(\mathcal{G}/(H, B))$ if and only if $I_{(K,S)}$ is a primitive ideal of $C^*(\mathcal{G})$. As Theorem 7.6 gives equivalent conditions for the primitivity of $I_{(K,S)}$, we conclude that:

Theorem 7.8. *Let $\mathcal{G}/(H, B)$ be a quotient ultragraph of \mathcal{G} . A gauge invariant ideal $J_{[K,S]}$ of $C^*(\mathcal{G}/(H, B))$ is primitive if and only if one of the following conditions holds:*

- (1) $S = B_K$, $\mathcal{G}^0 \setminus K$ satisfies the downward property, and every loop in $\mathcal{G} \setminus K$ has exits in $\mathcal{G} \setminus K$.
- (2) $S = B_K \setminus \{w\}$ for some $w \in B_K$, and $A \geq w$ for all $A \in \mathcal{G}^0 \setminus K$.

In particular, if $\mathcal{G}/(H, B)$ satisfies Condition (K), these conditions characterize all primitive ideals of $C^(\mathcal{G}/(H, B))$.*

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